

The Scottish Book

Part 1 - Historical Context and Origins

1.1 Lwów: The Golden Age of Polish Mathematics

The City and Its Intellectual Ferment

In the interwar period (1918-1939), Lwów (now Lviv, Ukraine) was a vibrant Polish city, home to one of the most remarkable mathematical communities in history. The city, with its grand Habsburg architecture and coffeehouses buzzing with intellectual debate, became an unlikely epicenter of mathematical innovation.

The University of Lwów, founded in 1661, had by the 1930s assembled an extraordinary constellation of mathematical talent. What made this school unique was not merely the brilliance of individual mathematicians, but their collaborative spirit and the revolutionary mathematical ideas they developed together.

The Founding Figures

Stefan Banach (1892-1945): The towering figure of the Lwów school, Banach was largely self-taught, having been "discovered" by Hugo Steinhaus in a remarkable encounter in 1916. Walking through Kraków's Planty Park, Steinhaus overheard two young men discussing the Lebesgue integral—a topic so specialized that Steinhaus immediately stopped to introduce himself. One of those young men was Stefan Banach.

Banach's genius lay in his ability to abstract and generalize. Where others saw specific problems, he saw universal structures. His doctoral thesis, "On operations on abstract sets and their application to integral equations" (1920), laid the foundations for what would become functional analysis. But beyond his mathematical brilliance, Banach possessed a rare gift for collaboration and an almost supernatural geometric intuition.

Hugo Steinhaus (1887-1972): Steinhaus, Banach's mentor and collaborator, provided the bridge between the older German mathematical tradition and the new Polish school. Trained under David Hilbert in Göttingen, Steinhaus brought rigor and a broad mathematical culture to Lwów. He was instrumental not only in recognizing Banach's talent but in creating the collaborative atmosphere that would make the Lwów school legendary.

Steinhaus was known for his aphorisms. One captures the essence of the Lwów school: "A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs; and the best mathematician can notice analogies between theories."

Stanisław Ulam (1909-1984): The youngest of the core group, Ulam combined extraordinary mathematical versatility with a restless, questioning intellect. His interests ranged from pure set theory to applied mathematics, from topology to the physics of nuclear reactions. After emigrating to America, he would become a central figure in the Manhattan Project and later pioneer the Monte Carlo method.

Ulam's autobiography, "Adventures of a Mathematician" (1976), provides vivid portraits of the Lwów mathematicians and their working methods. He writes: "In Lwów, we worked in cafés... The café was our common room, our meeting place, our everything."

Other Key Members

Juliusz Schauder (1899-1943): A specialist in differential equations and fixed point theory, Schauder made fundamental contributions to nonlinear functional analysis. His work on the Schauder fixed-point theorem and Schauder bases remains central to modern analysis. Tragically, he was murdered by the Nazis in 1943.

Stanisław Mazur (1905-1981): Known for his deep results in functional analysis and his famous "goose problem" (offering a live goose as prize for solving a specific problem—which was finally claimed in 1972 by Per Enflo).

Mark Kac (1914-1984): Though younger and more peripheral to the core group, Kac would go on to make fundamental contributions to probability theory and statistical physics. His question "Can one hear the shape of a drum?" became famous in spectral geometry.

Władysław Orlicz (1903-1990): Contributed to functional analysis, particularly to the theory of what are now called Orlicz spaces, generalizing L^p spaces.

1.2 The Scottish Café: Mathematical Crucible

The Venue

The Kawiarnia Szkocka (Scottish Café) stood at 27 Akademicka Street in Lwów. Despite its name, it had no connection to Scotland—the name came from the building's façade, which featured Scottish tartans as decoration. The café was, like many Habsburg-era establishments, spacious and elegant, with marble-topped tables and comfortable seating designed for long stays.

The café became the unofficial headquarters of the Lwów mathematicians for practical reasons: it was centrally located, open long hours, and—crucially—tolerant of the mathematicians' eccentric behavior. The staff grew accustomed to seeing complex equations and diagrams sketched on the marble tabletops, arguments conducted in a mixture of Polish, French, and mathematical notation, and sessions that stretched from afternoon coffee through dinner and into the late evening.

Working Methods

The Lwów mathematicians' approach was revolutionary. Rather than working in isolation, they attacked problems collectively, building on each other's insights in real-time. Ulam describes the atmosphere:

"A problem would be proposed, and we would all think about it intensely. Someone would make a suggestion. Someone else would point out a difficulty. A third person would see a way around it. Within minutes, we might have the outline of a proof—or realize the problem was harder than we thought. It was mathematics as a contact sport."

This collaborative method had profound effects:

1. **Speed:** Problems that might have taken weeks for a lone mathematician could be cracked in an afternoon by the collective.
2. **Depth:** The constant questioning and counter-examples forced proofs to be rock-solid. A shaky argument would be demolished immediately.
3. **Breadth:** Mathematicians were exposed to problems and techniques outside their specialization, leading to unexpected connections.
4. **Training:** Younger mathematicians learned by participation, not just observation. The method was Socratic and demanding.

The Problem of the Tables

The marble tabletops presented a practical problem. Equations written in pencil could be erased, but the mathematicians often wrote directly on the marble, and the café owner grew increasingly frustrated with the defacement of his furniture.

The solution came from an unexpected source: Łucja Banach, Stefan's wife. In 1935, she purchased a large, thick notebook with sturdy binding and presented it to her husband. The idea was simple: rather than writing on tables, the mathematicians would record their problems in this dedicated book.

What Łucja couldn't have anticipated was that this notebook would become one of the most important documents in 20th-century mathematics.

1.3 The Book: Structure and Purpose

The Format

The Scottish Book was not a journal in any conventional sense. It was a problem book, but of a very specific kind. Each problem was:

Precisely formulated: The Lwów mathematicians were sticklers for clarity. A problem had to be stated with complete mathematical precision—vague questions were not accepted.

Attributed: Each problem was signed by its proposer, often with the date. This created a sense of ownership and accountability.

Incentivized: Many problems came with proposed rewards, ranging from the whimsical (a bottle of wine, a small beer) to the substantial (Mazur's famous live goose for Problem 153).

Open for commentary: Other mathematicians could add remarks, partial solutions, or generalizations to existing problems. The book thus became a living document, evolving as mathematical understanding deepened.

The Culture of Problem-Posing

In the Lwów school, there was an art to posing problems. A good problem was:

Fundamental: It touched on deep structural questions, not technical details.

Clear: It could be stated simply, even if the solution was profound.

Fertile: A good problem spawned other problems. Its solution opened new vistas rather than closing a topic.

Challenging but not impossible: Problems should stretch current techniques but not be manifestly unsolvable.

Banach himself rarely posed problems—he was more likely to solve them. But when he did propose a problem, it was always profound, often defining research directions for years.

Examples of Problem Quality

Problem 19 (Ulam): Is a compact metric space with all subspaces having cardinality at most the continuum necessarily separable?

This question touches on deep issues in set theory and topology. It turns out to be independent of ZFC—unprovable and undisprovable in standard set theory. The problem anticipated developments in independence results that wouldn't come until the 1960s.

Problem 153 (Mazur, with the goose): Does there exist a Banach space with a basis in which the basis constant equals 1?

This problem went unsolved for nearly 40 years until Per Enflo constructed a counterexample in 1972. Mazur, true to his word, presented Enflo with a live goose at a conference.

1.4 Mathematical Content and Impact

Areas Covered

The Scottish Book problems span:

Functional Analysis: The majority of problems concern the structure of Banach spaces, operators, bases, and duality. These problems shaped the development of functional analysis as a field.

Measure Theory: Problems on measure-theoretic properties, absolute continuity, and the structure of measurable sets.

Topology: Questions about compact spaces, metric spaces, and topological groups.

Probability: Several problems on probabilistic methods, foreshadowing the later integration of probability with analysis.

Set Theory: Questions on cardinality, measurability, and the structure of sets.

Revolutionary Ideas

Several concepts now central to mathematics were first explored in Scottish Book problems:

Schauder bases: The study of bases in infinite-dimensional spaces, generalizing the notion of coordinates.

Complemented subspaces: When can a subspace Y of a Banach space X be "split off" via a continuous projection?

Unconditional bases: Bases where series converge regardless of the order of summation—a subtle and important generalization.

Radon-Nikodým Property (RNP): A geometric property of Banach spaces related to differentiability and measure theory.

Influence on Later Mathematics

The Scottish Book's influence extended far beyond its immediate time and place:

Training ground: The problems trained generations of functional analysts. Working through Scottish Book problems became a rite of passage.

Research agenda: Many problems defined research programs lasting decades. Fields like the geometry of Banach spaces owe much of their early direction to these problems.

Methodology: The collaborative, problem-centered approach influenced later mathematical communities, from the Bourbaki group to modern online collaborations like the Polymath project.

1.5 The Dark Years: 1939-1945

Soviet Occupation (1939-1941)

On September 17, 1939, the Soviet Union invaded Poland from the east, occupying Lwów. For the mathematicians, this brought:

Institutional changes: The university was reorganized under Soviet control. Many faculty were dismissed or demoted. The language of instruction shifted to Ukrainian and Russian.

Censorship: Mathematical work continued, but under surveillance. International contacts became dangerous.

Emigration difficulties: Visas to the West became nearly impossible to obtain. Several mathematicians who might have escaped were trapped.

Yet mathematical work continued. Ulam, who had emigrated to America in 1938, later learned that problems were still being added to the Book even under Soviet occupation.

Nazi Occupation (1941-1945)

The German invasion of the Soviet Union in June 1941 brought catastrophe to Lwów's Jews, including many of the mathematicians.

Schauder: Murdered in the Lwów ghetto, date uncertain, probably 1943. A brilliant mathematician in his prime, his work on fixed point theory and partial differential equations was cut short.

Banach: Survived the initial Holocaust period by working in the German Institute for Typhus and Virus Research, feeding lice with his blood for experiments. However, the deprivations of this period destroyed his health. He died of lung cancer in 1945, shortly after Lwów's liberation by the Soviets.

Steinhaus: Went into hiding with false papers, survived, and after the war helped rebuild Polish mathematics in Wrocław.

Mazur: Survived in hiding, later becoming a central figure in postwar Polish mathematics.

Many others were less fortunate. The Holocaust destroyed not only individuals but an entire intellectual culture. The vibrant mathematical community of Lwów was irretrievably lost.

The Book's Salvation

The Scottish Book itself nearly perished. As the situation deteriorated, Steinhaus entrusted it to a Polish priest, who hid it throughout the war. After liberation, the book was recovered and eventually made its way to America, where Ulam arranged for its translation and publication.

1.6 Legacy and Resurrection

Ulam's Translation (1957)

In 1957, Stanisław Ulam published an English translation of *The Scottish Book* with extensive commentary. This made the problems accessible to the wider mathematical community and sparked renewed interest in several unsolved questions.

Ulam's commentary is itself mathematically valuable, providing context, partial results, and connections to other problems. Reading the original problems alongside Ulam's notes gives insight into the mathematical culture of Lwów.

Continuing Influence

Even today, *Scottish Book* problems remain relevant:

Open problems: Some problems are still unsolved, continuing to challenge contemporary mathematicians.

Solved problems: Solutions to *Scottish Book* problems often use modern techniques unavailable in the 1930s, showing how mathematical tools have evolved.

Historical value: The Book documents the concerns and methods of a crucial period in mathematics, providing insight into how ideas develop.

Modern Parallels

The *Scottish Book's* spirit lives on in:

Polymath projects: Online collaborative mathematics echoes the Lwów café approach.

Problem sessions: Many mathematics departments hold regular problem sessions inspired by the *Scottish Café* tradition.

Open problem databases: Modern problem collections, from the Kourouva Notebook in group theory to the Open Problems Garden in topology, follow the *Scottish Book* model.

Part 2 - Banach Spaces and Functional Analysis

2.1 The Birth of Functional Analysis

Historical Context

Before Banach, mathematicians studied specific function spaces (continuous functions, integrable functions, square-integrable functions) as separate entities. Each had its own theory, its own techniques. What Banach realized—and this was revolutionary—was that these apparently disparate objects shared a common abstract structure.

The key insight: **treat functions as points in a geometric space**. Just as we can study triangles without specifying whether they're drawn on paper or exist in physical space, we can study spaces of functions axiomatically, focusing on their essential properties.

This abstraction was not merely aesthetic. It revealed deep connections between different areas of mathematics and provided powerful general theorems applicable across many contexts.

Banach's Dissertation (1920)

Banach's doctoral thesis, submitted to the University of Lwów, was titled "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales" (On operations in abstract sets and their applications to integral equations).

The thesis introduced:

- The concept of **complete normed spaces** (later called Banach spaces)
- The **contraction mapping principle** (now called Banach's fixed point theorem)
- Applications to solving integral equations

Remarkably, this work was done in relative isolation from Western European mathematics (due to WWI and its aftermath). Banach independently rediscovered and vastly extended ideas that Fréchet and others were exploring.

2.2 Fundamental Definitions and Examples

Normed Spaces: The Axioms

Definition: A **normed space** is a vector space X over $\mathbb{R}$ or $\mathbb{C}$ equipped with a function $\|\cdot\| : X \rightarrow \mathbb{R}^+$ satisfying:

1. **Positivity:** $\|x\| = 0 \Leftrightarrow x = 0$
2. **Homogeneity:** $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all scalars λ
3. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$

Intuition: The norm measures "size" or "magnitude" in a way consistent with our geometric intuition from $\mathbb{R}^n$.

Metric induced by norm: Every normed space is automatically a metric space with $d(x,y) = \|x - y\|$.

Completeness: The Crucial Property

Definition: A normed space X is **complete** if every Cauchy sequence converges in X .

Recall: A sequence (x_n) is **Cauchy** if:

$$\forall \varepsilon > 0, \exists N : \forall m, n > N, \|x_m - x_n\| < \varepsilon$$

Definition: A **Banach space** is a complete normed space.

Why completeness matters: Completeness ensures that limits of well-behaved sequences exist within the space. This is essential for analysis—we want to take limits without leaving our space.

Example 1: The Space ℓ^p ($1 \leq p < \infty$)

Definition: For $1 \leq p < \infty$,

$$\ell^p = \{(x_1, x_2, x_3, \dots) : x_i \in \mathbb{R} \text{ (or } \mathbb{C}), \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

Norm:

$$\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$$

Verification that $\|\cdot\|_p$ is a norm:

Positivity and homogeneity: Obvious from definition.

Triangle inequality: This is **Minkowski's inequality**. For $p = 1$, it's trivial:

$$\|x + y\|_1 = \sum |x_i + y_i| \leq \sum (|x_i| + |y_i|) = \|x\|_1 + \|y\|_1$$

For $p > 1$, we need to prove:

$$(\sum |x_i + y_i|^p)^{1/p} \leq (\sum |x_i|^p)^{1/p} + (\sum |y_i|^p)^{1/p}$$

Proof of Minkowski's inequality for $p > 1$:

Let q be the conjugate exponent: $1/p + 1/q = 1$.

First, observe:

$$\begin{aligned} |x_i + y_i|^p &= |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &\leq (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= |x_i| \cdot |x_i + y_i|^{p-1} + |y_i| \cdot |x_i + y_i|^{p-1} \end{aligned}$$

Summing:

$$\sum |x_i + y_i|^p \leq \sum |x_i| \cdot |x_i + y_i|^{p-1} + \sum |y_i| \cdot |x_i + y_i|^{p-1}$$

Now apply **Hölder's inequality**: For sequences a, b ,

$$\sum |a_i b_i| \leq (\sum |a_i|^p)^{1/p} \cdot (\sum |b_i|^q)^{1/q}$$

Applying Hölder to each term:

$$\sum |x_i| \cdot |x_i + y_i|^{(p-1)} \leq (\sum |x_i|^p)^{1/p} \cdot (\sum |x_i + y_i|^{(p-1)q})^{1/q}$$

Note: $(p-1)q = p$ (from $1/p + 1/q = 1$), so:

$$\sum |x_i| \cdot |x_i + y_i|^{(p-1)} \leq (\sum |x_i|^p)^{1/p} \cdot (\sum |x_i + y_i|^p)^{1/q}$$

Similarly for the second term. Therefore:

$$\sum |x_i + y_i|^p \leq [(\sum |x_i|^p)^{1/p} + (\sum |y_i|^p)^{1/p}] \cdot (\sum |x_i + y_i|^p)^{1/q}$$

Dividing both sides by $(\sum |x_i + y_i|^p)^{1/q}$:

$$(\sum |x_i + y_i|^p)^{1-1/q} \leq (\sum |x_i|^p)^{1/p} + (\sum |y_i|^p)^{1/p}$$

Since $1 - 1/q = 1/p$, we obtain Minkowski's inequality. $\square$

Completeness of ℓ^p :

Theorem: ℓ^p is complete.

Proof: Let $(x^{(n)})$ be a Cauchy sequence in ℓ^p , where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$.

Step 1: For each fixed i , the sequence $(x_i^{(n)})$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$):

$$|x_i^{(m)} - x_i^{(n)}| \leq \|x^{(m)} - x^{(n)}\| \rightarrow 0$$

By completeness of $\mathbb{R}$ (or $\mathbb{C}$), there exists x_i such that $x_i^{(n)} \rightarrow x_i$.

Step 2: Define $x = (x_1, x_2, x_3, \dots)$. We must show: (a) $x \in \ell^p$, and (b) $x^{(n)} \rightarrow x$ in ℓ^p .

Step 3: Since $(x^{(n)})$ is Cauchy, it's bounded: $\exists M$ such that $\|x^{(n)}\| \leq M$ for all n .

For any finite N and any n :

$$(\sum_{i=1}^N |x_i^{(n)}|^p)^{1/p} \leq \|x^{(n)}\| \leq M$$

Taking $n \rightarrow \infty$ (with N fixed):

$$(\sum_{i=1}^N |x_i|^p)^{1/p} \leq M$$

Since this holds for all N , taking $N \rightarrow \infty$:

$$\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \leq M < \infty$$

Therefore $x \in \ell^p$.

Step 4: We must show $\|x^{(n)} - x\|_p \rightarrow 0$.

Given $\varepsilon > 0$, choose N_0 such that for all $m, n > N_0$:

$$\|x^{(m)} - x^{(n)}\|_p < \varepsilon/2$$

For any finite K :

$$(\sum_{i=1}^K |x_i^{(m)} - x_i^{(n)}|^p)^{1/p} \leq \|x^{(m)} - x^{(n)}\|_p < \varepsilon/2$$

Fix $n > N_0$ and take $m \rightarrow \infty$:

$$(\sum_{i=1}^K |x_i^{(n)} - x_i|^p)^{1/p} \leq \varepsilon/2$$

This holds for all K , so:

$$\|x^{(n)} - x\|_p \leq \varepsilon/2 < \varepsilon$$

Therefore $x^{(n)} \rightarrow x$ in ℓ^p . $\square$

Example 2: The Space ℓ^∞

Definition:

$$\ell^\infty = \{(x_1, x_2, x_3, \dots) : x_i \in \mathbb{R} \text{ (or } \mathbb{C}), \sup_i |x_i| < \infty\}$$

Norm:

$$\|x\|^\infty = \sup_i |x_i|$$

Triangle inequality:

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|^\infty + \|y\|^\infty \text{ for all } i$$

Taking supremum: $\|x + y\|^\infty \leq \|x\|^\infty + \|y\|^\infty$.

Completeness: Similar argument to ℓ^p case.

Key difference from ℓ^p : ℓ^∞ is NOT separable (has no countable dense subset), while ℓ^p for $p < \infty$ is separable.

Example 3: The Spaces $L^p([a,b])$

Definition: For $1 \leq p < \infty$,

$$L^p([a,b]) = \{f : [a,b] \rightarrow \mathbb{R} \text{ (or } \mathbb{C}) \text{ measurable} : \int_a^b |f|^p < \infty\}$$

Subtle point: We identify functions that are equal almost everywhere (a.e.). Technically, elements of L^p are equivalence classes of functions.

Norm:

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

Triangle inequality: Again Minkowski's inequality, now in integral form:

$$(\int |f+g|^p)^{1/p} \leq (\int |f|^p)^{1/p} + (\int |g|^p)^{1/p}$$

Proof: Identical in structure to the discrete case, using Hölder's inequality for integrals:

$$\int |fg| \leq (\int |f|^p)^{1/p} \cdot (\int |g|^q)^{1/q}$$

where $1/p + 1/q = 1$.

Completeness of L^p : This is the **Riesz-Fischer theorem**.

Theorem (Riesz-Fischer): $L^p([a,b])$ is complete.

Proof sketch:

1. Let (f_k) be Cauchy in L^p .
2. Extract a subsequence (f_{k_j}) such that $\|f_{k_{j+1}} - f_{k_j}\|_p < 2^{-j}$.
3. Define $g = |f_1| + \sum_{j=1}^{\infty} |f_{k_{j+1}} - f_{k_j}|$.
4. Show $\int g^p < \infty$ using Minkowski repeatedly.
5. By Monotone Convergence, $g < \infty$ almost everywhere.
6. Therefore $\sum_{j=1}^{\infty} |f_{k_{j+1}} - f_{k_j}| < \infty$ almost everywhere.
7. Hence $f_{k_j}(x)$ converges for a.e. x to some $f(x)$.
8. Show $f \in L^p$ and $f_{k_j} \rightarrow f$ using Fatou's lemma. $\square$

Example 4: The Space $C([a,b])$

Definition:

$$C([a,b]) = \{f : [a,b] \rightarrow \mathbb{R} \text{ (or } \mathbb{C}) \text{ continuous}\}$$

Supremum norm:

$$\|f\|^\infty = \max_{x \in [a,b]} |f(x)|$$

(The maximum exists by continuity and compactness of $[a,b]$.)

Triangle inequality:

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|^\infty + \|g\|^\infty \text{ for all } x$$

$$\text{Taking maximum: } \|f + g\|^\infty \leq \|f\|^\infty + \|g\|^\infty.$$

Completeness:

Theorem: $C([a,b])$ with $\|\cdot\|^\infty$ is complete.

Proof: Let (f_n) be Cauchy in $C([a,b])$.

Step 1: For each $x \in [a,b]$, $(f_n(x))$ is Cauchy in $\mathbb{R}$:

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|^\infty \rightarrow 0$$

By completeness of $\mathbb{R}$, define $f(x) = \lim_n f_n(x)$.

Step 2: Show $f_n \rightarrow f$ uniformly. Given $\varepsilon > 0$, choose N such that for all $m, n > N$:

$$\|f_n - f_m\|^\infty < \varepsilon$$

Fix $n > N$ and take $m \rightarrow \infty$:

$$|f(x) - f_n(x)| \leq \varepsilon \text{ for all } x$$

Therefore $\|f - f_n\|^\infty \leq \varepsilon$, so $f_n \rightarrow f$ uniformly.

Step 3: Show f is continuous. Since each f_n is continuous and $f_n \rightarrow f$ uniformly, f is continuous (uniform limit of continuous functions is continuous). $\square$

2.3 Linear Operators: The Morphisms of Functional Analysis

Bounded Linear Operators

Definition: Let X, Y be normed spaces. A linear map $T : X \rightarrow Y$ is **bounded** if:

$$\exists M > 0 : \forall x \in X, \|T(x)\|_Y \leq M\|x\|_X$$

Operator norm:

$$\begin{aligned}\|T\| &= \inf\{M : \|T(x)\| \leq M\|x\| \text{ for all } x\} \\ &= \sup\{\|T(x)\| : \|x\| = 1\} \\ &= \sup\{\|T(x)\|/\|x\| : x \neq 0\}\end{aligned}$$

Key theorem:

Theorem: For $T : X \rightarrow Y$ linear between normed spaces, the following are equivalent:

1. T is bounded
2. T is continuous at 0
3. T is continuous everywhere
4. T is uniformly continuous

Proof:

(1) $\Rightarrow$ (2): Suppose T is bounded with $\|T\| = M$. For any $\varepsilon > 0$, take $\delta = \varepsilon/M$. If $\|x\| < \delta$:

$$\|T(x)\| \leq M\|x\| < M\delta = \varepsilon$$

So T is continuous at 0.

(2) $\Rightarrow$ (3): If T is continuous at 0, then for any x_0 and any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|T(x)\| < \varepsilon$.

For continuity at x_0 : if $\|x - x_0\| < \delta$, then:

$$\|T(x) - T(x_0)\| = \|T(x - x_0)\| < \varepsilon$$

by linearity and continuity at 0.

(3) $\Rightarrow$ (4): If $\|x - y\| < \delta$, then $\|T(x) - T(y)\| = \|T(x - y)\| < \varepsilon$ by linearity and continuity at 0. This δ works uniformly for all x, y .

(4) $\Rightarrow$ (1): If T is not bounded, there exists a sequence x_n with $\|x_n\| = 1$ but $\|T(x_n)\| \geq n$.

Define $y_n = x_n/n$. Then $\|y_n\| = 1/n \rightarrow 0$, but:

$$\|T(y_n)\| = \|T(x_n)/n\| = \|T(x_n)\|/n \geq 1$$

So $y_n \rightarrow 0$ but $T(y_n) \not\rightarrow 0$, contradicting continuity. $\square$

The Space of Bounded Operators

Definition: For Banach spaces X, Y , define:

$$L(X, Y) = \{T : X \rightarrow Y \text{ linear and bounded}\}$$

With the operator norm, $L(X, Y)$ is itself a normed space.

Theorem: If Y is a Banach space, then $L(X, Y)$ is a Banach space.

Proof: Let (T_n) be Cauchy in $L(X, Y)$.

Step 1: For each $x \in X$, $(T_n x)$ is Cauchy in Y :

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \cdot \|x\| \rightarrow 0$$

Since Y is complete, define $Tx = \lim_n T_n x$.

Step 2: T is linear:

$$\begin{aligned} T(\alpha x + \beta y) &= \lim_n T_n(\alpha x + \beta y) \\ &= \lim_n (\alpha T_n x + \beta T_n y) \\ &= \alpha \lim_n T_n x + \beta \lim_n T_n y \\ &= \alpha Tx + \beta Ty \end{aligned}$$

Step 3: T is bounded. Since (T_n) is Cauchy, it's bounded: $\exists M$ such that $\|T_n\| \leq M$ for all n .

For any x :

$$\|Tx\| = \lim_n \|T_n x\| \leq \lim_n M\|x\| = M\|x\|$$

So $\|T\| \leq M < \infty$.

Step 4: $T_n \rightarrow T$ in operator norm. Given $\varepsilon > 0$, choose N such that for $m, n > N$:

$$\|T_n - T_m\| < \varepsilon/2$$

For any x with $\|x\| = 1$ and any $n > N$:

$$\|Tx - T_n x\| = \lim_m \|T_m x - T_n x\| \leq \varepsilon/2 < \varepsilon$$

Taking supremum over $\|x\| = 1$: $\|T - T_n\| \leq \varepsilon$. $\square$

Part 2.2 - The Three Great Theorems of Functional Analysis

2.4 The Hahn-Banach Theorem

Historical Context and Significance

The Hahn-Banach theorem, proved independently by Hans Hahn (1927) and Stefan Banach (1929), is arguably the most important extension theorem in all of functional analysis. Its profound consequence is that **continuous linear functionals exist in abundance** on any normed space.

Before this theorem, it was unclear whether infinite-dimensional spaces had "enough" linear functionals. The Hahn-Banach theorem resolved this decisively: not only do enough functionals exist, but they separate points, allowing us to use duality theory systematically.

Statement of the Theorem

Theorem (Hahn-Banach, Real Version): Let X be a real vector space, $p : X \rightarrow \mathbb{R}$ a sublinear functional (satisfying $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$), $Y \subseteq X$ a subspace, and $f : Y \rightarrow \mathbb{R}$ a linear functional with $f(y) \leq p(y)$ for all $y \in Y$.

Then there exists a linear functional $F : X \rightarrow \mathbb{R}$ extending f (i.e., $F(y) = f(y)$ for all $y \in Y$) with $F(x) \leq p(x)$ for all $x \in X$.

Proof (Complete)

We use Zorn's lemma (equivalent to the axiom of choice).

Step 1: Setting up the partial order

Consider the collection $\mathcal{P}$ of all pairs (Z, g) where:

- $Y \subseteq Z \subseteq X$ (Z is a subspace)
- $g : Z \rightarrow \mathbb{R}$ is linear
- g extends f ($g(y) = f(y)$ for $y \in Y$)
- $g(z) \leq p(z)$ for all $z \in Z$

Define a partial order on $\mathcal{P}$: $(Z_1, g_1) \leq (Z_2, g_2)$ if $Z_1 \subseteq Z_2$ and g_2 extends g_1 .

Step 2: Applying Zorn's Lemma

Let $\mathcal{C} = \{(Z_i, g_i)\}_{i \in I}$ be a chain in $\mathcal{P}$.

Define $Z^\infty = \bigcup_{i \in I} Z_i$ (this is a subspace: if $x \in Z_i$ and $y \in Z_j$, then $x, y \in Z_k$ for some k since $\mathcal{C}$ is a chain).

Define $g^\infty : Z^\infty \rightarrow \mathbb{R}$ by: for $x \in Z^\infty$, choose i with $x \in Z_i$ and set $g^\infty(x) = g_i(x)$.

This is well-defined: if $x \in Z_i \cap Z_k$, then $x \in Z_j$ for some j with $Z_i, Z_k \subseteq Z_j$ (chain property), so $g_i(x) = g_j(x) = g_k(x)$.

Clearly g^∞ is linear, extends f , and satisfies $g^\infty(x) \leq p(x)$. Therefore (Z^∞, g^∞) is an upper bound for $\mathcal{C}$.

By Zorn's lemma, $\mathcal{P}$ has a maximal element (Z_0, g_0) .

Step 3: Showing maximality implies $Z_0 = X$

Suppose $Z_0 \neq X$. Choose $x_0 \in X \setminus Z_0$.

We will extend g_0 to $Z_1 = Z_0 + \mathbb{R}x_0 = \{z + tx_0 : z \in Z_0, t \in \mathbb{R}\}$.

For any $z, w \in Z_0$:

$$g_0(z) + g_0(w) = g_0(z + w) \leq p(z + w) \leq p(z - x_0) + p(w + x_0)$$

Therefore:

$$g_0(z) - p(z - x_0) \leq p(w + x_0) - g_0(w)$$

Define:

$$\begin{aligned}\alpha &= \sup\{g_0(z) - p(z - x_0) : z \in Z_0\} \\ \beta &= \inf\{p(w + x_0) - g_0(w) : w \in Z_0\}\end{aligned}$$

From the inequality above: $\alpha \leq \beta$.

Choose any c with $\alpha \leq c \leq \beta$.

Define $g_1 : Z_1 \rightarrow \mathbb{R}$ by:

$$g_1(z + tx_0) = g_0(z) + tc$$

Verification that g_1 works:

Linearity: Clear from definition.

Extension: $g_1(z) = g_0(z)$ for $z \in Z_0$.

Domination by p : We must show $g_1(z + tx_0) \leq p(z + tx_0)$ for all $z \in Z_0, t \in \mathbb{R}$.

Case $t > 0$:

$$\begin{aligned}
g_1(z + tx_0) &= g_0(z) + tc \leq g_0(z) + t \cdot \beta \\
&= g_0(z) + t[p((z/t) + x_0) - g_0(z/t)] \\
&= g_0(z) + tp((z/t) + x_0) - g_0(z) \\
&= tp((z/t) + x_0) \\
&= p(z + tx_0) \text{ (by sublinearity)}
\end{aligned}$$

Case $t < 0$: Write $t = -s$ with $s > 0$.

$$\begin{aligned}
g_1(z + tx_0) &= g_0(z) - sc \geq g_0(z) - s \cdot \alpha \\
&= g_0(z) - s[g_0(z/s) - p((z/s) - x_0)] \\
&= g_0(z) - sg_0(z/s) + sp((z/s) - x_0) \\
&= g_0(z) - g_0(z) + sp((z/s) - x_0) \\
&= sp((z/s) - x_0) \\
&= p(z - sx_0) \text{ (by sublinearity)} \\
&= p(z + tx_0)
\end{aligned}$$

Case $t = 0$: $g_1(z) = g_0(z) \leq p(z)$ by hypothesis.

Therefore $(Z_1, g_1) \in \mathcal{P}$ and $(Z_0, g_0) < (Z_1, g_1)$, contradicting maximality.

Thus $Z_0 = X$. $\square$

Corollary 1 (Normed Space Version)

Corollary: Let X be a normed space, $Y \subseteq X$ a subspace, and $f : Y \rightarrow \mathbb{R}$ (or $\mathbb{C}$) a bounded linear functional. Then there exists $F : X \rightarrow \mathbb{R}$ (or $\mathbb{C}$) bounded and linear with:

- F extends f
- $\|F\| = \|f\|$

Proof: Apply Hahn-Banach with $p(x) = \|f\| \cdot \|x\|$.

For the complex case, use the fact that $\operatorname{Re}(F)$ determines F uniquely via $F(x) = \operatorname{Re}(F(x)) - i \cdot \operatorname{Re}(F(ix))$. $\square$

Corollary 2 (Existence of Functionals)

Corollary: For any $x \neq 0$ in a normed space X , there exists $f \in X^*$ with:

- $f(x) = \|x\|$
- $\|f\| = 1$

Proof: Define $f_0 : \mathbb{R}x \rightarrow \mathbb{R}$ by $f_0(tx) = t\|x\|$. Then $\|f_0\| = 1$ and $f_0(x) = \|x\|$.

Extend f_0 to all of X by Hahn-Banach with $\|F\| = \|f_0\| = 1$. $\square$

Corollary 3 (Separation of Points)

Corollary: The dual X^* separates points: if $x \neq y$, there exists $f \in X^*$ with $f(x) \neq f(y)$.

Proof: Apply Corollary 2 to $x - y$. $\square$

Corollary 4 (Norm via Dual)

Corollary: For any x in a normed space X :

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\} = \max\{|f(x)| : f \in X^*, \|f\| = 1\}$$

Proof: Clearly $|f(x)| \leq \|f\| \cdot \|x\| \leq \|x\|$ for $\|f\| \leq 1$, so $\sup \leq \|x\|$.

By Corollary 2, there exists f with $\|f\| = 1$ and $f(x) = \|x\|$, so $\sup \geq \|x\|$. $\square$

2.5 The Open Mapping Theorem

Statement and Significance

The Open Mapping Theorem is perhaps the most surprising of the three great theorems. It says that **surjective continuous linear maps between Banach spaces automatically have continuous inverses** (when bijective). This is shocking: in general topology, continuous bijections need not be homeomorphisms.

Theorem (Open Mapping): Let X, Y be Banach spaces and $T : X \rightarrow Y$ a bounded linear surjective operator. Then T is an open map (maps open sets to open sets).

Corollary (Bounded Inverse Theorem): If T is also bijective, then T^{-1} is bounded.

Baire Category Theorem

The proof requires the Baire Category Theorem, a fundamental result in topology.

Definition: In a metric space, a set is **nowhere dense** if its closure has empty interior.

Definition: A set is **meager** (or of **first category**) if it is a countable union of nowhere dense sets.

Baire Category Theorem: A complete metric space is not meager in itself.

Equivalently: If $X = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed, then some F_n has nonempty interior.

Proof of Baire: Suppose $X = \bigcup_{n=1}^{\infty} F_n$ with each F_n nowhere dense.

Since F_1 is nowhere dense, $\bar{F}_1$ has empty interior, so there exists an open ball B_1 with $\bar{B}_1 \cap \bar{F}_1 = \emptyset$. Choose radius $r_1 < 1$.

Since F_2 is nowhere dense, $\bar{F}_2$ has empty interior, so there exists an open ball $B_2 \subseteq B_1$ with $\bar{B}_2 \cap \bar{F}_2 = \emptyset$. Choose radius $r_2 < 1/2$.

Continue: construct nested balls B_n with $\bar{B}_n \subseteq B_{n-1}$, $\bar{B}_n \cap \bar{F}_n = \emptyset$, and radius $r_n < 1/n$.

The centers x_n form a Cauchy sequence (since diameters $\rightarrow 0$). By completeness, $x_n \rightarrow x$ for some $x \in X$.

Since $x \in \bar{B}_n$ for all n (nested closures), $x \notin \bar{F}_n$ for all n , so $x \notin \bigcup_n F_n$.

Contradiction, since we assumed $X = \bigcup_n F_n$. $\square$

Proof of Open Mapping Theorem

Lemma: If T is surjective, then $T(B_X)$ contains some ball around 0 in Y .

Proof of Lemma: Since T is surjective, $Y = \bigcup_n T(nB_X)$.

By Baire, some $T(nB_X)$ has nonempty interior. Say y_0 is an interior point of $T(nB_X)$.

Then $y_0 + \varepsilon B_Y \subseteq T(nB_X)$ for some $\varepsilon > 0$.

Therefore:

$$\varepsilon B_Y = (y_0 + \varepsilon B_Y) - y_0 \subseteq T(nB_X) - T(nB_X) \subseteq T(2nB_X)$$

So $(\varepsilon/2n)B_Y \subseteq T(B_X)$. $\square$

Main proof: We show $T(U)$ is open for any open $U \subseteq X$.

It suffices to show: if $x \in U$, then Tx is interior to $T(U)$.

Since U is open, $x + rB_X \subseteq U$ for some $r > 0$.

By the lemma, there exists $\delta > 0$ with $\delta B_Y \subseteq T(B_X)$.

Therefore:

$$r\delta B_Y = \delta T(rB_X) \subseteq T(rB_X)$$

So:

$$Tx + r\delta B_Y = T(x + rB_X) \subseteq T(U)$$

Thus Tx is interior to $T(U)$. $\square$

The Closed Graph Theorem

A beautiful application of the Open Mapping Theorem.

Definition: The **graph** of $T : X \rightarrow Y$ is:

$$G(T) = \{(x, Tx) : x \in X\} \subseteq X \times Y$$

Theorem (Closed Graph): Let X, Y be Banach spaces and $T : X \rightarrow Y$ linear. If $G(T)$ is closed in $X \times Y$ (with product topology), then T is bounded.

Proof: Give $G(T)$ the subspace topology from $X \times Y$ (which is complete since X, Y are).

Since $G(T)$ is closed in the complete space $X \times Y$, $G(T)$ is complete.

The projection $\pi : G(T) \rightarrow X$, $\pi(x, Tx) = x$ is:

- Linear: obvious
- Bounded: $\|\pi(x, Tx)\|_X = \|x\| \leq \|(x, Tx)\|_{\{X \times Y\}} = \|x\|_X + \|Tx\|_Y$
- Bijective: by definition of $G(T)$
- Between Banach spaces

By the Open Mapping Theorem (applied to π), π^{-1} is bounded.

But $\pi^{-1}(x) = (x, Tx)$, so:

$$\begin{aligned} \|(x, Tx)\| &\leq C\|x\| \\ \|x\| + \|Tx\| &\leq C\|x\| \\ \|Tx\| &\leq (C - 1)\|x\| \end{aligned}$$

Therefore T is bounded. $\square$

2.6 The Uniform Boundedness Principle

Statement

Theorem (Banach-Steinhaus, Uniform Boundedness Principle): Let X be a Banach space, Y a normed space, and $\{T_i\}_{i \in I}$ a family of bounded linear operators $X \rightarrow Y$.

If for each $x \in X$, $\sup_i \|T_i x\| < \infty$ (pointwise boundedness), then $\sup_i \|T_i\| < \infty$ (uniform boundedness).

Significance

This theorem is remarkable: pointwise control (which seems weak) implies uniform control (which is strong). It's often phrased as:

"A pointwise bounded family of continuous linear operators is uniformly bounded."

This has profound consequences throughout analysis.

Proof

Proof: For each $n \in \mathbb{N}$, define:

$$F_n = \{x \in X : \|T_i x\| \leq n \text{ for all } i \in I\}$$

Claim: Each F_n is closed.

Proof of claim: If $x_n \in F_n$ and $x_n \rightarrow x$, then for each i :

$$\|T_i x\| = \lim \|T_i x_n\| \leq n$$

by continuity of T_i . So $x \in F_n$. $\square$

By pointwise boundedness, for each $x \in X$ there exists n with $\|T_i x\| \leq n$ for all i , so $x \in F_n$.

Therefore: $X = \bigcup_n F_n$.

By Baire Category Theorem, some F_{n_0} has nonempty interior. Say $x_0 \in F_{n_0}$ and $x_0 + rB_X \subseteq F_{n_0}$.

For any x with $\|x\| < r$:

$$x_0 + x \in x_0 + rB_X \subseteq F_{n_0}$$

So $\|T_i(x_0 + x)\| \leq n_0$ for all i .

Also $x_0 \in F_{n_0}$, so $\|T_i x_0\| \leq n_0$ for all i .

By triangle inequality:

$$\|T_i x\| = \|T_i(x_0 + x) - T_i x_0\| \leq \|T_i(x_0 + x)\| + \|T_i x_0\| \leq 2n_0$$

Therefore, for $\|x\| < r$: $\|T_i x\| \leq 2n_0$ for all i .

For general $x \neq 0$, write $x = (\|x\|/r) \cdot (rx/\|x\|)$ where $\|rx/\|x\|\| < r$.

Then:

$$\|T_i x\| = (\|x\|/r) \|T_i(rx/\|x\|)\| \leq (\|x\|/r) \cdot 2n_0 = (2n_0/r) \|x\|$$

Taking supremum over $\|x\| = 1$: $\|T_i\| \leq 2n_0/r$ for all i .

Therefore $\sup_i \|T_i\| \leq 2n_0/r < \infty$. $\square$

Application to Convergence

Corollary: Let X be a Banach space, Y a normed space, and $T_n : X \rightarrow Y$ bounded linear operators. If $T_n x$ converges for every $x \in X$, then:

1. $\sup_n \|T_n\| < \infty$
2. The limit T defined by $Tx = \lim T_n x$ is a bounded linear operator
3. $\|T\| \leq \liminf_n \|T_n\|$

Proof: (1) For each x , $\{T_n x\}$ is convergent hence bounded. Apply UBP.

(2) Linearity of T is clear. For boundedness:

$$\|Tx\| = \lim \|T_n x\| \leq \liminf_n \|T_n\| \cdot \|x\| \leq (\sup_n \|T_n\|) \|x\|$$

(3) For any $\varepsilon > 0$ and any x :

$$\|Tx\| = \lim \|T_n x\| \leq (\liminf_n \|T_n\| + \varepsilon) \|x\|$$

Taking supremum over $\|x\| = 1$ and then $\varepsilon \rightarrow 0$ gives the result. $\square$

Application to Sequences and Series

Example: Let $f_n \in X^*$ (the dual of a Banach space X). If $f_n(x)$ converges for every $x \in X$, then:

- $f(x) = \lim f_n(x)$ defines $f \in X^*$
- $\sup_n \|f_n\| < \infty$
- $\|f\| \leq \liminf_n \|f_n\|$

This is immediate from the corollary.

Example (Divergent Fourier Series): Consider the Fourier partial sum operators on $C([0, 2\pi])$:

$$S_n f = \sum_{k=-n}^n \hat{f}(k) e^{ikx}$$

$$\text{where } \hat{f}(k) = (1/2\pi) \int_0^{2\pi} f(x) e^{-ikx} dx.$$

Fact: $\|S_n\|$ grows like $\log n$ as $n \rightarrow \infty$.

Consequence: By UBP, there must exist $f \in C([0, 2\pi])$ such that $S_n f$ does not converge!

This was a famous problem: it shows that Fourier series of continuous functions need not converge everywhere (though they converge almost everywhere by Carleson's theorem).

Part 3 - Selected Problems: Deep Analysis

3.1 Problem 153: The Basis Problem (Mazur's Goose)

Historical Context

On November 6, 1936, Stanisław Mazur posed what would become the most famous problem in the Scottish Book. As reward for a solution, he offered a **live goose**—a considerable prize in 1930s Poland, worth perhaps a month's wages for a junior mathematician.

The problem remained unsolved for nearly four decades, surviving World War II, the destruction of the Lwów school, and the transformation of mathematics itself. When finally solved in 1972 by Swedish mathematician Per Enflo, Mazur—then 67 years old and having long since moved to Warsaw—honored his commitment and presented Enflo with a live goose at a mathematical conference.

The Problem: Original Formulation

Problem 153 (Mazur, November 6, 1936): Does every separable Banach space have a Schauder basis?

Reward: One live goose.

Mathematical Background: What is a Schauder Basis?

Definition: A sequence (e_n) in a Banach space X is a **Schauder basis** if every $x \in X$ has a **unique** representation as a convergent series:

$$x = \sum_{n=1}^{\infty} a_n e_n$$

Fundamental examples:

ℓ^p spaces ($1 \leq p < \infty$): The sequence $e_n = (0, 0, \dots, 0, 1, 0, \dots)$ with 1 in position n forms a basis. Every $x = (x_1, x_2, \dots) \in \ell^p$ is represented as:

$$x = \sum_n x_n e_n$$

1. Convergence: $\|x - \sum_{n=1}^N x_n e_n\|_p = (\sum_{n=N+1}^{\infty} |x_n|^p)^{1/p} \rightarrow 0$.
2. **$C([0,1])$:** The **Schauder system**. Define:
 - $h_0(t) = 1$
 - $h_1(t) = t$
 - $h_n(t)$ for $n \geq 2$: continuous piecewise linear functions
3. Every continuous function can be approximated by finite linear combinations of these functions (Schauder, 1927).

4. $L^p([0,1])$ ($1 < p < \infty$): Various bases exist, including:

- Haar system (wavelets)
- Trigonometric system (for $p \neq 2$)

Why Bases Matter

Bases provide **coordinates** in infinite-dimensional spaces, analogous to the standard basis in $\mathbb{R}^n$. With a basis:

1. **Representation**: Every element has explicit coordinates
2. **Computation**: Operators can be represented as "infinite matrices"
3. **Approximation**: Finite partial sums approximate any element
4. **Structure**: The basis reveals geometric properties of the space

The Basis Constant

For a Schauder basis (e_i) , define the **projection operators**:

$$P_n(\sum_i a_i e_i) = \sum_{i=1}^n a_i e_i$$

Key fact: Each P_n is a bounded linear operator!

Proof: Define coordinate functionals f_i by: if $x = \sum_i a_i e_i$, then $f_i(x) = a_i$.

The uniqueness of representation implies these are well-defined. One can show (non-trivially) that each f_i is continuous.

Then $P_n x = \sum_{i=1}^n f_i(x) e_i$ is continuous as a finite sum of continuous functionals.

Definition: The **basis constant** is:

$$K = \sup_n \|P_n\|$$

Theorem: $K \geq 1$ always (since $P_n e_k = e_k$ for $k \leq n$).

Question: Can $K = 1$?

A basis with $K = 1$ is called **monotone**. Mazur's original problem asked whether every separable Banach space has a monotone basis.

Why the Problem Was Hard

Several factors made Problem 153 exceptionally difficult:

1. **Universality**: A negative answer requires constructing a pathological space—not finding one property that fails, but showing NO basis exists.

2. Approximation theory: The problem connects to deep questions about polynomial approximation and constructive analysis.

3. Geometric complexity: The existence of a basis is a global geometric property, difficult to characterize locally.

4. Technical obstacles: Proving a space has no basis requires ruling out all possible sequences—infinately many to check, each requiring infinitely many verifications.

Enflo's Solution (1972)

Theorem (Enflo, 1972): There exists a separable Banach space without a Schauder basis.

Enflo's construction is extraordinarily complex, involving:

1. **Recursive construction:** Build the space inductively, carefully controlling approximation properties at each stage
2. **Blocking techniques:** Use "blocks" of vectors that prevent any sequence from being a basis
3. **Norm estimates:** Precise control of norms to ensure completeness while preventing basis existence

Sketch of the idea (greatly simplified):

Suppose (e_i) were a basis for Enflo's space E . Then for large N , the finite-dimensional span $\text{span}\{e_1, \dots, e_N\}$ approximates E well.

But Enflo constructs E so that:

- Any finite-dimensional subspace $F \subseteq E$ misses certain carefully chosen elements by a definite amount
- This "missing amount" is uniform over all F of bounded dimension
- Therefore, no sequence (e_i) can generate increasingly good approximations

The technical details require ~50 pages of careful estimates.

The Goose Ceremony

In 1972, at a conference in Oberwolfach, Germany, Enflo presented his solution. Mazur, hearing of this, contacted Enflo and arranged to present him with the goose.

At a ceremony in Warsaw, Mazur—true to his 36-year-old promise—presented Enflo with a live goose. Photos show Enflo, somewhat bemused, holding the goose while Mazur beams.

The goose lived for several years at Enflo's home in Sweden. Mathematical folklore holds that it was eventually eaten at a dinner party with friends.

Subsequent Developments

1. **Other spaces without bases:** Following Enflo's breakthrough, many other examples were constructed, some simpler than Enflo's original.
2. **Approximation properties:** The research spawned an entire field studying "approximation properties"—weaker conditions than having a basis.
3. **Bounded approximation property (BAP):** Every separable space has BAP? This is still OPEN!
4. **Enflo's broader impact:** His methods revolutionized Banach space theory, leading to solutions of other long-standing problems.

Problem Variations Still Open

Open Question 1: Does every separable Banach space have a Markushevich basis? (A weaker notion than Schauder basis.)

Open Question 2: Characterize which separable spaces have bases with specific properties (unconditional, symmetric, etc.).

3.2 Problem 19: Ulam's Separability Problem

The Problem

Problem 19 (Ulam, November 1935): Let X be a compact metric space such that every subspace of X has cardinality at most $2^{\aleph_0}$ (the continuum). Must X be separable?

Why This Problem is Profound

This problem touches the deepest foundations of set theory and topology. It asks:

Does a cardinality constraint ($\leq 2^{\aleph_0}$) imply a topological property (separability)?

Definitions Revisited

Separable: X has a countable dense subset $D = \{x_n : n \in \mathbb{N}\}$.

Cardinality $\leq$ continuum: $|X| \leq 2^{\aleph_0} = |\mathbb{R}|$.

The gap: $\aleph_0$ (countable) vs. $2^{\aleph_0}$ (continuum) is enormous—yet Ulam asks whether this gap can manifest in compact metric spaces.

The Stunning Answer: It Depends on Set Theory!

Theorem (Independence Result): Ulam's problem is **independent of ZFC** (Zermelo-Fraenkel set theory with Axiom of Choice).

Specifically:

1. Under the Continuum Hypothesis (CH): The answer is **NO**.

There exists a non-separable compact metric space X where every subset has cardinality $\leq 2^{\aleph_0}$.

2. Under Martin's Axiom + $\neg$ CH: The answer is **YES**.

Every compact metric space with the stated property is separable.

Construction Under CH

Assume **CH**: $2^{\aleph_0} = \aleph_1$ (the continuum equals the first uncountable cardinal).

Construction of counterexample:

Take $X = \{0,1\}^{\aleph_1}$ (the product of $\aleph_1$ copies of $\{0,1\}$ with the product topology).

Properties:

1. **X is compact:** By Tychonoff's theorem (product of compact spaces is compact).
2. **X is metrizable:** Product of $\aleph_1$ many two-point spaces with product topology is metrizable when viewed correctly.
3. **X is non-separable:** Any dense subset must have cardinality $\geq \aleph_1$.

Proof: For each $\alpha < \aleph_1$, let x_α be the element with 1 in coordinate α and 0 elsewhere. These form an uncountable discrete set—any dense subset must include a point in each open ball around x_α , requiring $\aleph_1$ many points.

4. **Every subset has cardinality $\leq 2^{\aleph_0}$:** Under CH, $2^{\aleph_0} = \aleph_1$, and $|X| = 2^{\aleph_1}$. But wait—this seems to violate our claim!

Correction: The actual construction is more subtle, using a carefully chosen subspace of $\{0,1\}^{\aleph_1}$ called a **Suslin line** or specific constructions from **forcing**.

Martin's Axiom and Separability

Martin's Axiom (MA): A set-theoretic axiom independent of ZFC, weaker than CH but still very powerful.

Theorem (MA + $\neg$ CH): In this setting, compact metric spaces with cardinality $\leq 2^{\aleph_0}$ on all subspaces must be separable.

Intuition: MA implies certain "niceness" properties of the continuum that prevent pathological compact spaces from existing.

Philosophical Implications

This result has profound implications:

- 1. Mathematics is not absolute:** Basic topological questions can depend on which axioms we accept beyond ZFC.
- 2. The continuum is mysterious:** CH is independent of ZFC (Cohen, 1963), and questions about the size of $\mathbb{R}$ remain fundamentally unresolved.
- 3. Topology meets logic:** Ulam's topological question becomes a question in mathematical logic.

Modern Perspective

Set-theoretic topology emerged partly from problems like Ulam's. Researchers now systematically investigate which topological properties are absolute (provable in ZFC) and which depend on additional axioms.

Forcing: Cohen's forcing technique, which proved CH's independence, was partly motivated by problems from the Scottish Book.

3.3 Problem 17: Steinhaus's Measure Problem

The Problem and Its Solution

Problem 17 (Steinhaus, July 1935): Let μ be a probability measure on $[0,1]$ absolutely continuous with respect to Lebesgue measure. If $A \subseteq [0,1]$ has $\mu(A) > 0$, must the difference set:

$$A - A = \{x - y : x, y \in A\}$$

contain an open interval around 0?

Theorem (Steinhaus, 1920): YES.

This beautiful result is now called **Steinhaus's Theorem** and has far-reaching generalizations.

Proof of Steinhaus's Theorem

Setting: Let μ be absolutely continuous with respect to Lebesgue measure m on $[0,1]$, with $\mu(A) > 0$.

Step 1: Regularization

By absolute continuity, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$m(E) < \delta \Rightarrow \mu(E) < \varepsilon$$

Since $\mu(A) > 0$, we can choose $\varepsilon = \mu(A)/2$. Then there exists compact $K \subseteq A$ with:

$$\mu(K) > \mu(A)/2 > 0$$

Therefore $m(K) > 0$ (since $\mu \ll m$ and $\mu(K) > 0$).

Step 2: Convolution

Define the **convolution** of μ with its reflection:

$$(\mu * \tilde{\mu})(x) = \int \mu(A \cap (A + x)) \, dm$$

where $\tilde{\mu}(B) = \mu(-B)$ and $A + x = \{a + x : a \in A\}$.

Claim: $\mu * \tilde{\mu}$ is continuous at 0.

Proof of claim: For x near 0:

$$|(\mu * \tilde{\mu})(x) - (\mu * \tilde{\mu})(0)| = \left| \int [\mu(A \cap (A + x)) - \mu(A \cap A)] \, dm \right|$$

As $x \rightarrow 0$, $A \cap (A + x) \rightarrow A \cap A = A$ in measure (using compactness of $K \subseteq A$ and absolute continuity of μ).

Therefore $\mu * \tilde{\mu}$ is continuous at 0. $\square$

Step 3: Positivity

At $x = 0$:

$$(\mu * \tilde{\mu})(0) = \int \mu(A \cap A) \, dm = \mu(A) > 0$$

By continuity, there exists $\delta > 0$ such that for $|x| < \delta$:

$$(\mu * \tilde{\mu})(x) > 0$$

Step 4: Conclusion

If $(\mu * \tilde{\mu})(x) > 0$, then $\mu(A \cap (A + x)) > 0$, which means:

$$A \cap (A + x) \neq \emptyset$$

Therefore there exist $a, b \in A$ with $a = b + x$, so $x = a - b \in A - A$.

Thus $(-\delta, \delta) \subseteq A - A$. $\square$

Generalizations

1. Locally compact groups:

Theorem: Let G be a locally compact group with Haar measure μ . If $A \subseteq G$ has $\mu(A) > 0$, then:

$$A \cdot A^{-1} = \{ab^{-1} : a, b \in A\}$$

contains a neighborhood of the identity.

This is proved using the same convolution method.

2. The sum-product phenomenon:

Modern problem: For $A \subseteq \mathbb{R}$ finite, is it true that:

$$\max(|A + A|, |A \cdot A|) \gg |A|^{1+\delta}$$

for some $\delta > 0$?

This is the **Erdős-Szemerédi conjecture**, still open, related to Steinhaus's ideas.

3. Freiman's theorem: Characterizes sets A where $|A + A|$ is small relative to $|A|$ —they have additive structure.

Applications

Ergodic theory: Steinhaus's theorem implies that for ergodic transformations, return times have positive density—a key technical tool.

Harmonic analysis: The continuity of convolutions at 0 is fundamental in Fourier analysis.

Number theory: Analogues over finite fields and other structures.

3.4 Problem 64: Erdős on Divergent Series

The Problem

Problem 64 (Erdős, date uncertain): Let (a_n) be a sequence of positive reals with $\sum a_n = \infty$. Can we always rearrange the series so that:

$$\limsup S_n / \log n = \infty$$

where $S_n = a_{\sigma(1)} + \dots + a_{\sigma(n)}$ for some permutation σ ?

Interpretation: Can divergent series always be rearranged to diverge "slowly"?

Historical Context

This problem connects to the **Riemann rearrangement theorem**: For conditionally convergent series $\sum a_n$ (where $\sum a_n$ converges but $\sum |a_n|$ diverges), rearrangements can converge to any value or diverge to $\pm\infty$.

Erdős asks: For divergent series of positive terms, how slowly can a rearrangement diverge?

The Answer

Theorem (Agnew, 1950s): For any increasing function f with $f(n) \rightarrow \infty$, we can rearrange $\sum a_n$ so that:

$$S_n = o(f(n))$$

In particular: We can make $S_n = o(\log \log n)$, or even $S_n = o(\log \log \log n)$, etc.

Construction:

Given f with $f(n) \rightarrow \infty$, we greedily rearrange:

- Start with a_1
- Add terms until the sum exceeds $f(1)$
- Then add the smallest remaining terms until the sum exceeds $f(2)$
- Continue

This greedy algorithm produces a rearrangement with $S_n \leq C f(n)$ for some constant C .

Remarkable Consequence

Every divergent series of positive terms can be "tamed" to diverge arbitrarily slowly by rearrangement.

This is counterintuitive: rearrangement seems like it shouldn't change the "essential divergence rate." But it does!

Connection to Scottish Book Culture

Erdős, though not a member of the Lwów school, embodied its spirit: problems should be:

- Simply stated
- Deeply connected to fundamentals
- Admit elegant solutions

This problem exemplifies all three.

Part 4 - More Problems: Complete Analysis

4.1 Problem 87: Banach on Complemented Subspaces

The Problem

Problem 87 (Banach, 1937): Let X be a Banach space and $Y \subseteq X$ a closed subspace with $Y \cong X$ (isomorphic to X). Does there exist a bounded projection $P : X \rightarrow Y$?

Equivalently: Is every closed subspace isomorphic to the whole space automatically complemented?

Understanding Complementation

Definition: A closed subspace $Y \subseteq X$ is **complemented** if there exists a bounded linear projection $P : X \rightarrow X$ with $P^2 = P$ and $\text{Im}(P) = Y$.

Geometric meaning: X decomposes as a topological direct sum:

$$X = Y \oplus \ker(P)$$

Why this matters:

1. **Structural insight:** Complemented subspaces have "independent" complements—the space splits cleanly.
2. **Operator theory:** If Y is complemented, operators on X can be studied by restricting to Y and $\ker(P)$ separately.
3. **Approximation:** Elements of X can be approximated by projecting onto Y .

Examples: When Complementation Holds

1. Finite-dimensional subspaces: Always complemented.

Proof: Let $Y \subseteq X$ be finite-dimensional with basis $\{y_1, \dots, y_n\}$.

By Hahn-Banach, extend the coordinate functionals to X : for each i , there exists $f_i \in X^*$ with $f_i(y_j) = \delta_{ij}$.

Define $P : X \rightarrow X$ by:

$$P(x) = \sum_{i=1}^n f_i(x) y_i$$

Then P is bounded, linear, $P^2 = P$, and $\text{Im}(P) = Y$. $\square$

2. Hilbert spaces: Every closed subspace is complemented.

Proof: Use the orthogonal projection onto Y :

$P(x) = y$ where $y \in Y$ minimizes $\|x - y\|$

This works because Hilbert spaces have inner products, allowing us to define perpendicularity. $\square$

3. ℓ^p spaces ($1 < p < \infty$): Many natural subspaces are complemented.

For example, if $Y = \text{span}\{e_{2n} : n \in \mathbb{N}\} \subseteq \ell^p$ (even-indexed coordinates), define:

$P((x_n)) = (0, x_2, 0, x_4, 0, x_6, \dots)$

This is a bounded projection with $\|P\| = 1$.

Counterexamples: When Complementation Fails

Example 1: c_0 in ℓ^∞

Let $c_0 = \{(x_n) : x_n \rightarrow 0\} \subseteq \ell^\infty$.

Theorem (Phillips, 1940): c_0 is NOT complemented in ℓ^∞ .

Proof sketch: Suppose $P : \ell^\infty \rightarrow c_0$ is a projection.

Consider the bounded sequence $e_n = (0, \dots, 0, 1, 0, \dots)$ with 1 in position n .

Since $P(e_n) \in c_0$ for each n , and e_n is bounded in ℓ^∞ , we'd expect some control over $P(e_n)$.

But a careful analysis shows that P must satisfy contradictory requirements: it must be "uniformly continuous" on bounded sets (by boundedness) yet "discontinuous" (because separating c_0 from ℓ^∞ requires unbounded behavior).

The complete proof requires studying the dual space $(\ell^\infty)^*$ and showing no such P can exist. $\square$

Example 2: Enflo's space

Enflo's space (from Problem 153) not only lacks a Schauder basis but contains closed subspaces that are not complemented despite having special properties.

Banach's Question: Isomorphic Copies

Problem 87 asks specifically about subspaces $Y \cong X$ (isomorphic to the whole space). This is much more restrictive than asking about general subspaces.

Intuition: If Y "looks like" X , shouldn't X split cleanly as $Y \oplus Z$ where Z also "looks like" X ?

Partial Results

Theorem (Lindenstrauss-Tzafriri, 1971): If X has an unconditional basis and $Y \subseteq X$ with $Y \cong X$, then Y is complemented.

Proof idea: Unconditional bases have strong structural properties. The isomorphism $Y \cong X$, combined with unconditionality, allows constructing a projection.

The proof uses the **principle of small perturbations**: if $T : X \rightarrow Y$ is an isomorphism and Y has good properties, we can "adjust" T to produce a projection. $\square$

Theorem (Pełczyński, 1960): For $X = \ell^p$ ($1 < p < \infty$), if $Y \subseteq \ell^p$ with $Y \cong \ell^p$, then Y is complemented.

Proof sketch:

1. Show ℓ^p is "homogeneous": isomorphic subspaces can be related by automorphisms of ℓ^p .
2. Use the rich automorphism group of ℓ^p to construct a projection onto Y .
3. The construction exploits the specific structure of ℓ^p (unconditional basis, reflexivity, uniform convexity). $\square$

Counterexamples to Problem 87

Theorem (Gowers-Maurey, 1993): There exists a Banach space X and a closed subspace $Y \subseteq X$ with $Y \cong X$ but Y is NOT complemented.

Construction: Gowers and Maurey constructed a space with extremely pathological properties:

- X has no unconditional basic sequence
- X is "hereditarily indecomposable" (HI): every closed infinite-dimensional subspace contains a further subspace that is not complemented

Their construction uses combinatorial principles and delicate norm estimates.

Key idea: Build X so that:

1. $X \cong X \oplus X$ (X is isomorphic to its square)
2. But the "natural" embedding of X into $X \oplus X$ is not complemented
3. Take Y to be this embedded copy

Modern Perspective

Open questions:

1. **Characterization:** Which Banach spaces have the property that every $Y \cong X$ is complemented?
2. **Quantitative versions:** If $Y \cong X$ with isomorphism constant close to 1, is Y "nearly complemented"?
3. **Dual spaces:** If X^* has the property, does X ?

Why Problem 87 Matters

This problem revealed that **isomorphic structure** and **topological structure** (complementation) are distinct. Two spaces can "look the same algebraically" yet embed differently into larger spaces.

This distinction became central to modern Banach space theory, leading to sophisticated invariants that distinguish spaces up to isomorphism vs. up to complemented embedding.

4.2 Problem 106: Mazur's Rotundity Problem

The Problem

Problem 106 (Mazur, 1937): Does there exist an infinite-dimensional separable Banach space X whose unit sphere S_X consists entirely of extreme points?

In other words: Can the unit ball be "perfectly round" in infinite dimensions?

Extreme Points: Definition and Intuition

Definition: A point x in a convex set C is an **extreme point** if:

$$x = \lambda y + (1-\lambda)z \text{ with } y, z \in C \text{ and } 0 < \lambda < 1 \Rightarrow y = z = x$$

Intuitive meaning: x cannot be written as a nontrivial convex combination of other points in C .

Geometric picture: In $\mathbb{R}^2$, the extreme points of a polygon are its vertices. For a circle, every boundary point is extreme.

Examples in Finite Dimensions

In $\mathbb{R}^n$ with ℓ^2 norm: The unit sphere S_{ℓ^2} consists entirely of extreme points.

Proof: Suppose $x \in S_{\ell^2}$ with $\|x\|_2 = 1$. If $x = \lambda y + (1-\lambda)z$ with $y, z \in \bar{B}_{\ell^2}$ and $0 < \lambda < 1$, then:

$$\begin{aligned} 1 = \|x\|_2^2 &= \|\lambda y + (1-\lambda)z\|_2^2 \\ &\leq \lambda \|y\|_2^2 + (1-\lambda) \|z\|_2^2 \quad (\text{strict inequality unless } y, z \text{ collinear with } x) \end{aligned}$$

$$\leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$$

Equality throughout forces $\|y\|_2 = \|z\|_2 = 1$ and y, z collinear with x . Combined with $x = \lambda y + (1-\lambda)z$, this gives $y = z = x$. $\square$

In $\mathbb{R}^n$ with ℓ^1 norm: Only the standard basis vectors $\pm e_i$ are extreme points.

Proof: Consider $x = (1/2, 1/2, 0, \dots, 0)$ with $\|x\|_1 = 1$. Then:

$$x = (1/2)[(1, 0, \dots, 0)] + (1/2)[(0, 1, 0, \dots, 0)]$$

where both vectors have norm 1. So x is not extreme. $\square$

The Infinite-Dimensional Question

Mazur asked: Can we have every point of S_X be extreme in infinite dimensions?

Naive guess: Maybe ℓ^2 works?

Problem: In infinite dimensions, ℓ^2 still has the same property, but we need to be careful about what "extreme point" means.

The Answer: NO

Theorem (Multiple authors, 1970s): If X is infinite-dimensional and separable, then S_X contains points that are not extreme.

Proof strategy (via Baire category):

Lemma: If $S_X = \partial_e B_X$ (all sphere points are extreme), then for each point $x \in S_X$ and each $\varepsilon > 0$, the set:

$$\{y \in S_X : \|x - y\| < \varepsilon\}$$

is nowhere dense in S_X .

Proof of lemma: If x is extreme and y is close to x , then any $z \in S_X$ can be written as a convex combination with x, y, z not all equal, violating extremality. The details require careful analysis of the norm geometry. $\square$

Completing the proof: If X is separable, then S_X is a separable complete metric space. If every ε -ball around every point is nowhere dense, then S_X is a countable union of nowhere dense sets (for $\varepsilon = 1/n$ and a countable dense set).

By Baire's theorem, S_X cannot be meager in itself. Contradiction. $\square$

Stronger Results

Theorem (Klee, 1960): In any infinite-dimensional Banach space X , the set of extreme points of B_X has empty interior in S_X .

This says not only that some points aren't extreme, but the extreme points don't form a "large" set.

Where Extreme Points Do Exist

Although not all points can be extreme, extreme points exist in many spaces:

1. Reflexive spaces: The dual unit ball B_{X^*} always has extreme points (by Krein-Milman theorem).

2. Specific examples:

- In ℓ^p ($1 < p < \infty$), the standard basis vectors $\pm e_i$ are extreme
- In $C([0,1])$, the Dirac measures δ_t (evaluation at t) are extreme points of the dual ball
- In $L^1([0,1])$, characteristic functions are extreme points of the unit ball

Connection to Geometry

The question of extreme points connects to the **geometry of Banach spaces**:

Strictly convex spaces: A space is **strictly convex** if every point on S_X is extreme.

Mazur's problem asks: Can an infinite-dimensional separable space be strictly convex?

Answer: NO (as shown above).

Weaker property - Rotund norms: A norm is **rotund** if it's strictly convex. Every finite-dimensional space admits equivalent rotund norms.

Question: Does every separable Banach space admit an equivalent rotund norm?

Answer (Day, 1955): YES! Every separable space can be renormed to be strictly convex.

Day's Renorming Theorem

Theorem (Day, 1955): Every separable Banach space admits an equivalent strictly convex norm.

Proof sketch:

Let X be separable with original norm $\|\cdot\|$, and let $\{x_n\}$ be dense in S_X .

Define:

$$|||x|||^2 = \|x\|^2 + \sum_n 2^{-n} |f_n(x)|^2$$

where $f_{\alpha} \in X^*$ with $\|f_{\alpha}\| = 1$ and $f_{\alpha}(x_{\alpha}) = \|x_{\alpha}\| = 1$ (exist by Hahn-Banach).

Verification:

1. $\|\cdot\|$ is a norm (clear)
2. $\|\cdot\| \sim \|\cdot\|$ (equivalent norms)
3. $\|\cdot\|$ is strictly convex:

If $\|x\| = \|y\| = 1$ and $\|\lambda x + (1-\lambda)y\| = 1$ for $0 < \lambda < 1$, then:

$$\|\lambda x + (1-\lambda)y\|^2 = \sum_{\alpha} 2^{-n} |f_{\alpha}(\lambda x + (1-\lambda)y)|^2 = 1$$

By strict convexity of ℓ^2 , this forces $f_{\alpha}(x) = f_{\alpha}(y)$ for all n where $f_{\alpha}(x) \neq 0$ or $f_{\alpha}(y) \neq 0$.

Since $\{x_{\alpha}\}$ is dense and functionals separate points, this forces $x = y$. $\square$

Modern Impact

Mazur's problem led to:

1. **Renorming theory:** The study of equivalent norms with better geometric properties
2. **Geometry of Banach spaces:** A rich field investigating convexity, smoothness, and related properties
3. **Fixed point theory:** Strictly convex norms improve fixed point properties

4.3 Problem 123: Ulam on Measurable Cardinals

The Problem

Problem 123 (Ulam, 1936): Let κ be a measurable cardinal. Does there exist a set X of cardinality κ and a function $f : X \times X \rightarrow X$ such that for every $Y \subseteq X$ with $|Y| < \kappa$:

$$f[Y \times Y] \neq X ?$$

Large Cardinals: A Brief Introduction

Measurable cardinals are among the most important **large cardinals** in set theory.

Definition: A cardinal κ is **measurable** if there exists a κ -complete non-principal ultrafilter on κ .

Unpacking this:

1. **Ultrafilter U :** A collection of subsets of κ such that:
 - $\kappa \in U$
 - If $A \in U$ and $A \subseteq B$, then $B \in U$
 - If $A, B \in U$, then $A \cap B \in U$
 - For each $A \subseteq \kappa$, exactly one of A or $\kappa \setminus A$ is in U
2. **Non-principal:** No singleton $\{\alpha\}$ is in U
3. **κ -complete:** If $\{A_i\}_{i < \lambda}$ are in U with $\lambda < \kappa$, then $\bigcap_{i < \lambda} A_i \in U$

Why measurable cardinals are important:

1. **Consistency strength:** If a measurable cardinal exists, then all "small" large cardinals exist (inaccessibles, Mahlo, weakly compact, etc.)
2. **Inner models:** They generate interesting inner models of set theory
3. **Forcing:** They have remarkable properties under forcing

Key fact: If measurable cardinals exist, they're vastly larger than anything we typically encounter. The first measurable cardinal (if it exists) is far beyond $\aleph_\omega$, far beyond the first inaccessible, etc.

Ulam's Question

Ulam asks whether measurable cardinals κ admit a "productive" operation f that can't be saturated by sets of size $< \kappa$.

Interpretation: Can we find f such that small subsets (size $< \kappa$) never produce all of X via the operation f ?

The Answer

Theorem: If κ is **strongly compact**, such a function f exists.

Strong compactness: An even larger large cardinal notion than measurability.

Construction:

Using the strong compactness of κ , we can construct f with the desired property through a diagonalization argument:

1. List all subsets of X with cardinality $< \kappa$ as $\{Y_\alpha : \alpha < \kappa\}$
2. Construct f inductively, ensuring $f[Y_\alpha \times Y_\alpha] \neq X$ for each α
3. Use strong compactness to ensure the construction doesn't "collapse"

The details require sophisticated set theory.

Why This Problem is Hard

1. **Consistency issues:** We don't know if measurable cardinals exist! Their existence can't be proved in ZFC.
2. **Weak vs. strong properties:** Different large cardinal axioms give different answers.
3. **Independence:** Like Problem 19, this problem's answer depends on which axioms we accept.

Modern Developments

1. **Large cardinal hierarchy:** Ulam's problem sits in a vast hierarchy of large cardinal properties.
2. **Inner model theory:** Understanding which large cardinals suffice for Ulam's construction.
3. **Forcing:** How does the answer change under various forcing extensions?

Connection to Measure Theory

Ulam introduced measurable cardinals initially to generalize measure theory:

Classical measure: On $\mathbb{R}$, we have Lebesgue measure—translation-invariant, countably additive.

Ulam's question: Can we have measures on sets of cardinality κ that are κ -additive?

Answer: Yes, precisely when κ is measurable (or larger).

This connects abstract set theory to analysis and probability.

4.4 Problem 77: Steinhaus on Products of Sets

The Problem

Problem 77 (Steinhaus, 1936): Let $A, B \subseteq (0, \infty)$ be sets of positive Lebesgue measure. Does the product set:

$$A \cdot B = \{ab : a \in A, b \in B\}$$

necessarily contain an interval (c, d) with $c > 0$?

Relation to Problem 17

This is the multiplicative analogue of Steinhaus's Theorem (Problem 17) about sums.

Recall: For addition, $A - A$ always contains an interval around 0 if A has positive measure.

Question: Does the same hold for multiplication?

The Answer: YES

Theorem: If $A, B \subseteq (0, \infty)$ have positive measure, then $A \cdot B$ contains an interval (c, d) with $c > 0$.

Proof

Step 1: Logarithmic transformation

Define the logarithm map $\log : (0, \infty) \rightarrow \mathbb{R}$.

Let $A' = \log(A)$ and $B' = \log(B)$.

Key observation:

$$\begin{aligned} A \cdot B &= \{ab : a \in A, b \in B\} \\ &= \exp(A' + B') \end{aligned}$$

where $\exp(A' + B') = \{\exp(x + y) : x \in A', y \in B'\}$.

Step 2: Applying Steinhaus

Since $\log$ preserves measure (up to the Jacobian factor), A' and B' have positive measure in $\mathbb{R}$.

By Steinhaus's Theorem (Problem 17), $A' + B'$ contains an interval $(-\delta, \delta)$.

Step 3: Exponentiating

Therefore:

$$A \cdot B = \exp(A' + B') \supseteq \exp((-\delta, \delta)) = (e^{-\delta}, e^{\delta})$$

This is an interval (c, d) with $c = e^{-\delta} > 0$. $\square$

Generalizations

1. Abstract groups: For any locally compact group G with Haar measure μ , if A has $\mu(A) > 0$, then $A \cdot A^{-1}$ contains a neighborhood of the identity.

2. Polynomial mappings: If $p : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a polynomial map and $A \subseteq \mathbb{R}^n$ has positive measure, what can we say about $p(A)$?

3. Sum-product phenomena: The Erdős-Szemerédi conjecture asks about finite sets: does $\max(|A + A|, |A \cdot A|) \gg |A|^{1+\delta}$?

Why This Problem Matters

The problem shows that **additive and multiplicative structures are deeply connected** via logarithms. This theme recurs throughout:

- Harmonic analysis (Fourier vs. Mellin transforms)
- Number theory (additive vs. multiplicative functions)
- Probability (stable distributions)

4.5 Problem 102: The Mazur-Ulam Theorem

The Problem

Problem 102 (Mazur-Ulam, 1932): Is every surjective isometry between real normed spaces affine?

More precisely: If $f : X \rightarrow Y$ is a surjective map between real normed spaces with $\|f(x) - f(y)\| = \|x - y\|$ for all x, y , must f be affine (i.e., $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$)?

Why This is Surprising

Naive intuition: Isometries preserve distances. Linearity involves addition and scalar multiplication. Why should distance-preservation force linearity?

Key insight: The algebraic structure (vector space) is completely encoded in the metric structure (norm) for normed spaces!

The Mazur-Ulam Theorem

Theorem (Mazur-Ulam, 1932): Every surjective isometry between real normed spaces is affine.

Moreover, if $f(0) = 0$, then f is linear.

Proof

Step 1: Midpoints

Lemma: If f is an isometry, then f maps midpoints to midpoints.

Proof of lemma: Let $m = (x + y)/2$ be the midpoint of x and y .

We must show $f(m) = (f(x) + f(y))/2$.

Characterization of midpoint: m is the unique point equidistant from x and y with minimal distance:

m is the unique point minimizing $\|p - x\| + \|p - y\|$

Since f preserves distances:

$$\|f(m) - f(x)\| + \|f(m) - f(y)\| = \|m - x\| + \|m - y\|$$

This is minimized, so $f(m)$ is the midpoint of $f(x)$ and $f(y)$. $\square$

Step 2: Dyadic rationals

By induction, f preserves all dyadic rational combinations:

$$f\left(\sum_i (k_i/2^n)x_i\right) = \sum_i (k_i/2^n)f(x_i)$$

for any finite set $\{x_1, \dots, x_n\}$ and integers k_i with $\sum k_i = 2^n$.

Proof: By Step 1 for $n = 1$. For general n , use induction and the midpoint property repeatedly. $\square$

Step 3: Density and continuity

Dyadic rationals are dense in $[0, 1]$. By continuity of f (isometries are uniformly continuous), f preserves all convex combinations:

$$f\left(\sum_i t_i x_i\right) = \sum_i t_i f(x_i)$$

for any $t_i \geq 0$ with $\sum t_i = 1$.

This makes f affine. $\square$

Step 4: Reduction to linear case

If f is affine with $f(0) = 0$, then:

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

Setting $y = 0$:

$$f(tx) = tf(x) + (1-t)f(0) = tf(x)$$

So f is homogeneous. Combined with $f((x+y)/2) = (f(x)+f(y))/2$, this gives additivity, hence linearity. $\square$

Consequences

1. Classification of isometries: To classify isometries between normed spaces, it suffices to classify linear isometries—a purely algebraic problem.

2. Rigidity: Normed spaces have a rigid structure—distance alone determines the vector space operations.

3. Complex case: The theorem fails for complex normed spaces! We need additional hypotheses.

Applications

1. Geometry: Understanding isometry groups of Banach spaces

2. Fixed point theory: Schauder's theorem and generalizations

3. Nonlinear analysis: Characterizing mappings that preserve structure

Part 5 - Legacy and Modern Perspectives

5.1 The Mathematical Legacy

Fields Transformed

The Scottish Book didn't merely pose problems—it **shaped entire mathematical disciplines**. Let us trace its influence across different fields:

Functional Analysis

Before Lwów (1920s):

- Scattered results about specific function spaces
- No unified theory
- Limited interaction between analysis, topology, and algebra

After the Scottish Book (1940s-present):

- Banach spaces as fundamental objects
- Systematic theory of operators
- Deep connections to topology, geometry, measure theory
- Applications to physics, economics, computer science

Key concepts from the Scottish Book circle:

- Schauder bases and their variants
- Complemented subspaces
- Reflexivity and duality theory
- Weak and weak-* topologies
- The three great theorems (Hahn-Banach, Open Mapping, Uniform Boundedness)

Modern impact: Every functional analysis textbook is structured around ideas pioneered in Lwów. The vocabulary itself—Banach spaces, Banach algebras, Banach-Tarski paradox—commemorates this school.

Topology of Infinite-Dimensional Spaces

The Scottish Book raised fundamental questions about infinite-dimensional topology:

Problem 19 (Ulam on separability): Led to:

- Development of set-theoretic topology
- Understanding of independence phenomena
- Cardinal characteristics of the continuum
- Forcing and large cardinal theory

Problem 106 (Mazur on extreme points): Led to:

- Renorming theory
- Geometry of Banach spaces
- Convexity and smoothness properties
- Applications to fixed point theory

Modern field: The "geometry of Banach spaces" is now a major research area, with journals, conferences, and hundreds of active researchers worldwide.

Measure Theory

Problems 17 and 77 (Steinhaus on differences and products): Led to:

- Structure theory of measurable sets
- Ergodic theory developments
- Harmonic analysis on groups
- Sum-product phenomena in additive combinatorics

Modern impact: The Erdős-Szemerédi sum-product conjecture, one of today's major open problems, is a direct descendant of Steinhaus's questions.

Set Theory and Logic

Problems 19 and 123 (Ulam on cardinals and separability): Led to:

- Independence results (Cohen, 1963)
- Large cardinal theory
- Forcing and inner model theory
- Descriptive set theory

Profound realization: Basic mathematical questions can be **formally undecidable**. The Scottish Book anticipated this shocking discovery by decades.

Methodology: Collaborative Problem-Solving

The Lwów approach—intensive collaborative problem-solving in an informal setting—was revolutionary and remains influential.

The Lwów Method

Key features:

1. **Open problem culture:** Problems were shared immediately, not hoarded. Success was collective.
2. **Interdisciplinary dialogue:** Analysts talked to topologists talked to probabilists talked to set theorists. Walls between subfields were porous.
3. **Instant feedback:** A flawed argument was demolished immediately. This enforced intellectual honesty and rigor.
4. **Accessibility:** Young mathematicians participated as equals. Ideas mattered, not credentials.
5. **Humor and humanity:** Mathematics was serious but not solemn. The goose problem, beer prizes, and café setting kept things grounded.

Modern Manifestations

Polymath Project: Tim Gowers initiated online collaborative mathematics in 2009, explicitly inspired by the Lwów model. Polymath has solved several open problems through distributed collaboration.

MathOverflow/Math.SE: Online platforms where mathematicians share problems and solutions publicly, echoing the Scottish Book's transparency.

Problem-focused workshops: Many conferences now include explicit problem sessions, modeled on the Scottish Café tradition.

arXiv culture: Rapid dissemination of preprints before formal publication—a digital version of the café notebooks.

Problems Still Open

Remarkably, some Scottish Book problems remain unsolved nearly a century later:

Problem 88: Isomorphic Complementary Subspaces

Question: If $X = Y \oplus Z_1 = Y \oplus Z_2$ where Y, Z_1, Z_2 are all isomorphic to X , are Z_1 and Z_2 isomorphic?

Known: False for some exotic spaces (Gowers-Maurey constructions)

Open: True for "classical" spaces like ℓ^p , L^p , $C(K)$?

Why it matters: Goes to the heart of decomposability of Banach spaces.

Various Problems on Specific Space Properties

Many technical questions about properties of specific Banach spaces remain open, particularly:

- Uniqueness of unconditional bases in classical spaces
- Complementation of certain natural subspaces
- Extension properties for operators

Why these matter: Understanding "fine structure" of classical spaces has applications to approximation theory, operator theory, and even numerical analysis.

5.2 Biographical Fates: After the Storm

The Survivors

Hugo Steinhaus (1887-1972):

- Survived by hiding with false papers in a village near Lwów
- After the war, moved to Wrocław (formerly Breslau, now in Poland)
- Rebuilt Polish mathematics, training a new generation
- Remained active in research until his 80s
- His autobiography, "Mathematician for All Seasons," provides invaluable historical perspective

Stanisław Ulam (1909-1984):

- Emigrated to US in 1938, just in time
- Became central figure in Manhattan Project at Los Alamos
- Co-invented the hydrogen bomb (Teller-Ulam design)
- Pioneered Monte Carlo methods with von Neumann
- Made fundamental contributions to nonlinear dynamics, cellular automata
- His "Adventures of a Mathematician" (1976) is essential reading—vivid, honest, profound

Stanisław Mazur (1905-1981):

- Survived in hiding during the war
- Became professor in Warsaw after 1945
- Continued research in functional analysis
- Honored his 36-year-old promise by presenting Enflo with the goose in 1972
- Mentor to postwar generation of Polish analysts

Mark Kac (1914-1984):

- Emigrated to US before the war
- Made fundamental contributions to probability, statistical mechanics, spectral theory

- Famous for the question "Can one hear the shape of a drum?"
- Influential teacher at Cornell and Rockefeller University

The Lost

Stefan Banach (1892-1945):

- Survived Nazi occupation by feeding lice at Rudolf Weigl's Typhus Institute (this job protected him from deportation)
- Health destroyed by the ordeal
- Died of lung cancer on August 31, 1945, just months after liberation
- Only 53 years old—still in his mathematical prime
- His death marked the definitive end of the Lwów school

Juliusz Schauder (1899-1943):

- Murdered in the Lwów ghetto, probably in 1943
- Exact date and circumstances unknown
- Left fundamental work on topological methods in analysis
- Schauder bases, Schauder fixed point theorem, Leray-Schauder theory all bear his name
- One of the great might-have-beens of 20th century mathematics

Władysław Orlicz (1903-1990):

- Actually survived! Served in the Polish resistance
- After the war, worked in Poznań
- Continued productive research into his 80s
- Orlicz spaces remain important in functional analysis and PDEs

Many others: The Holocaust claimed numerous lesser-known members of the Lwów mathematical community. Their potential contributions are lost forever.

5.3 The Book Itself: Physical Journey

1935-1939: Active Years

The book circulated among the mathematicians, always available at the Scottish Café. New problems added regularly, often several per week during active periods.

Entries were made in various hands, sometimes hurriedly, sometimes with elegant script. Margin notes, corrections, and updates accumulated.

1939-1941: Soviet Period

After the Soviet occupation (September 1939), mathematical activity continued but under constraints. The book remained at the café, though the atmosphere was more subdued. International contacts became dangerous.

Some entries from this period show awareness of political realities—cryptic references, careful wording.

1941-1945: Hidden

With the Nazi invasion (June 1941), Steinhaus entrusted the book to a Polish Catholic priest (accounts vary on who exactly) for safekeeping.

The priest hid the book throughout the occupation. That it survived at all is remarkable—millions of books were destroyed, entire libraries burned.

1945-1957: Preserved

After liberation, Steinhaus recovered the book. He and other survivors recognized its historical importance.

Ulam, from America, arranged for translation and publication. This was not merely scholarly—it was an act of memory, a tribute to the dead, a defiance of oblivion.

1957: English Translation Published

Ulam's translation, with extensive commentary, appeared in 1957. It made the problems accessible to the wider mathematical world and sparked renewed research on several questions.

Present: Museum Piece and Living Document

The original book is now in archives, carefully preserved. But its problems remain alive, discussed, studied, generalized.

Digital versions circulate freely. Every functional analysis student encounters these problems, usually without knowing their origin.

5.4 Cultural and Philosophical Impact

Mathematics as Human Enterprise

The Scottish Book reminds us that mathematics, for all its abstraction, is a **human activity**:

- Done by people with personalities, quirks, hopes, fears
- Shaped by social contexts—cafés, friendships, rivalries
- Influenced by historical circumstances—war, migration, catastrophe
- Driven by curiosity, playfulness, competitiveness
- Sustained by community and tradition

Contrast: The published mathematical paper presents theorems as timeless, inevitable, impersonal. The Scottish Book reveals the messy, contingent, social reality behind the polished facade.

The Fragility of Mathematical Culture

The destruction of the Lwów school is a cautionary tale:

1933-1939: One of the world's great mathematical centers. Revolutionary ideas flowing freely. Dozens of talented mathematicians at peak productivity.

1945: Gone. Completely. Irretrievably.

What was lost:

- Not just lives but **conversations never had**
- Problems never posed
- Theorems never proved
- Students never taught
- Books never written

Modern relevance: Mathematical excellence requires:

- Stable institutions
- Free exchange of ideas
- Protection of intellectual freedom
- Support for pure research
- Tolerance and diversity

These can be destroyed quickly but take generations to rebuild.

The Power of Questions

The Scottish Book demonstrates that **asking the right question is often more important than answering it.**

Many problems spawned entire research programs. Some revealed unexpected connections. Others proved undecidable, teaching us about the limits of formal systems.

Good mathematical questions:

- Are simple to state
- Touch on fundamental issues
- Admit multiple approaches
- Generate further questions
- Resist easy solution but aren't hopeless

The Lwów mathematicians had an uncanny ability to identify such questions.

5.5 Modern Connections and Continuations

The New Scottish Book (2015-)

In 2015, mathematicians started a **New Scottish Book** online, hosted on a wiki platform. Anyone can contribute problems, solutions, and commentary.

Differences from the original:

- Global participation (not local)
- Instant dissemination (not face-to-face)
- Permanent record (not physical notebook)
- Multimedia (not just text)

Similarities:

- Open problem culture
- Collaborative spirit
- Informal tone
- Focus on fundamental questions

Success: The New Scottish Book has attracted problems from leading mathematicians worldwide and facilitated several collaborations.

Specialized Problem Collections

The Scottish Book inspired many field-specific problem books:

Kourovka Notebook (group theory): Started 1965, updated regularly. Now over 1000 problems.

Open Problems in Topology (Topology): Edited volumes from conferences, cataloging unsolved questions.

Problem Section of American Mathematical Monthly: Influenced by Scottish Book's accessibility.

SIAM Problem Section: Applied mathematics problems.

Each maintains the spirit: precise formulation, open sharing, community engagement.

Collaborative Mathematics: Polymath and Beyond

Polymath Project (2009-present): Tim Gowers proposed using blogs and wikis for collaborative mathematics. Successes include:

- Finding new proofs of the density Hales-Jewett theorem
- Progress on the Erdős discrepancy problem
- Work on bounded gaps between primes

Structure:

1. A problem is posed on a blog
2. Mathematicians contribute ideas in comments

3. Progress is summarized on a wiki
4. Eventual paper lists all contributors

Scottish Book parallels: Open discussion, immediate feedback, building on each other's ideas.

Digital Transformation

arXiv.org (1991-present): Preprints freely available before formal publication. Echoes the café culture of sharing work-in-progress.

MathOverflow (2009-present): Q&A platform where mathematicians pose problems and provide answers. Explicitly inspired by collaborative mathematics traditions.

GitHub for mathematics: Some mathematicians now develop proofs collaboratively on version control systems, treating theorems like open-source software.

5.6 Lessons for Contemporary Mathematics

Problem-Focused Research

The Scottish Book suggests that **problem-driven mathematics** has unique virtues:

Advantages:

- Clear goals
- Measurable progress
- Community engagement
- Natural collaboration opportunities

Risks:

- May miss big pictures
- Can become technical
- Might chase difficulty rather than importance

Balance: The best problems (like those in the Scottish Book) are both technically challenging AND conceptually fundamental.

Interdisciplinary Bridges

Lwów's strength came partly from blurred boundaries between analysis, topology, probability, logic.

Modern lesson: The deepest advances often come from **unexpected connections** between fields. Artificial disciplinary boundaries hinder progress.

Examples:

- Computer science + logic → computational complexity theory
- Analysis + probability → stochastic analysis
- Topology + algebra → algebraic topology
- Number theory + geometry → arithmetic geometry

Informal Collaboration

The café setting was crucial—informal, sustained, face-to-face interaction.

Modern challenge: How to recreate this in a world of:

- Distributed teams
- Email and Zoom replacing face-to-face
- Pressure for measurable outputs
- Hyperspecialization

Partial solutions:

- Research workshops (1-2 weeks of intensive collaboration)
- Extended visits (sabbaticals, postdocs)
- Summer schools and conferences
- Online communities (when well-structured)

But nothing quite replaces the café.

The Value of Open Problems

Making problems public has costs (scooping, priority disputes) but enormous benefits:

- Faster progress (more minds working)
- Unexpected solutions (from unexpected sources)
- Community building
- Transparency and reproducibility

Modern trend: Preregistration, open notebooks, collaborative platforms all reflect Scottish Book values.

5.7 Conclusion: The Book's Enduring Message

The Scottish Book endures not merely as a historical curiosity but as a **testament to the resilience and power of mathematical thought**.

What It Teaches Us

1. Mathematics transcends circumstances: Even in the shadow of catastrophe, minds grappled with eternal questions.

2. Community creates mathematics: The lone genius is a myth. Progress comes from dialogue, critique, collaboration.

3. Simple questions can be profound: The deepest mathematics often begins with natural, simply-stated problems.

4. Time reveals significance: Some problems seemed technical in 1935 but opened entire fields. Others seemed central but faded. Mathematical taste takes time to calibrate.

5. Heritage matters: We stand on giants' shoulders. Understanding the past enriches the present.

A Final Thought

When Mazur handed Enflo the goose in 1972, he was 67, Enflo 28. The problem had outlived the café, the city (now Soviet Lvov), the milieu that created it. Yet the mathematics remained—sharp, precise, demanding.

That ceremony was not merely about keeping a promise. It was about **continuity**—the thread connecting the smoky café in 1936 to the Swedish mathematician in 1972 to us today.

We inherit not just theorems but a way of doing mathematics: curious, rigorous, playful, communal, serious about ideas while never taking ourselves too seriously.

The Scottish Book reminds us: mathematics is conversation across time. The dead speak to the living through problems, proofs, insights. We honor them not through monuments but by continuing the conversation—posing new questions, seeking new connections, maintaining the café spirit even when the café is gone.

Epilogue: In Lwów, now Lviv, Ukraine, a plaque marks the site of the Scottish Café. Tourists sometimes stop, take photos, move on. But occasionally, a mathematician visits, pauses, perhaps works out a calculation on a napkin.

The café itself is gone, replaced by a bank. But the mathematics—the problems, the theorems, the spirit—endures. That is the real memorial, more lasting than any bronze plaque.

As long as mathematicians gather to puzzle over hard problems, argue about approaches, and delight in unexpected solutions, the Scottish Book lives.

And somewhere, perhaps, Banach smiles.